Heptagonal quasicrystal tilings

1996 J. Phys. A: Math. Gen. 29 6877

(http://iopscience.iop.org/0305-4470/29/21/018)

J. Phys. A: Math. Gen. **29** (1996) 6877–6879. Printed in the UK

Heptagonal quasicrystal tilings

Juan García-Escudero
Departamento de Física, Universidad de Oviedo, 33007 Oviedo, Spain

Received 8 July 1996

Abstract. Four-colour aperiodic selfsimilar patterns in two dimensions with seven-fold symmetry and Bragg spectrum are introduced and described with the help of substitutional sequences.

Quasicrystals are solids with an essentially discrete diffraction diagram (Bragg peaks) showing non-crystallographic symmetries [1]. In order to study their structure with the help of tiling models it is necessary therefore to look for tilings with Bragg spectrum.

Context-free grammars and D0L systems have been used recently by the author for the description of two-colour planar patterns with non-crystallographic symmetries less than or equal to eight [2, 3] and icosahedral Danzer tilings [4]. The patterns with seven-fold symmetry obtained by Danzer [5, 1] have no discrete part in their diffraction spectra [4].

In this work we introduce seven-fold symmetry patterns which could be used for structure models of quasicrystals (this symmetry has not yet been found in diffraction experiments). They have been obtained by inflation and can be described as word sequences along the lines of [2, 3].

A D0L system (see [6]) is a triple $G = \{\Sigma, h, \omega\}$ where Σ is an alphabet, h is an endomorphism defined on the set Σ^* of all the words over the alphabet Σ, and ω referred to as the axiom is an element of Σ^*. The word sequence $E(G)$ generated by G consists of the words $h^0(\omega) = \omega, h(\omega), h^2(\omega), h^3(\omega), \ldots$ and the language of G is defined by $L(G) = \{h^i(\omega)/i \geqslant 0\}$. In what follows the endomorphism h will be defined by listing the productions for each letter.

The following example shows a selfsimilar tiling of the line obtained by interpreting the letters $X \in \Sigma$ as segments with appropriate lengths l_X. They correspond to the edges of the elementary two-dimensional tiles.

Consider the alphabet $\Sigma = \{U, V, W\}$, axiom: $\omega = U$ and set of production rules:

$$U \longmapsto UVW \qquad V \longmapsto UV \qquad W \longmapsto U. \tag{1}$$

The geometric interpretation of the word sequences is clear: if two letters follow one another, the corresponding segments will appear together in the structure. If we choose the segment lengths: $l_U/l_W = \beta = 1 + 2\cos(2\pi/7)$, $l_V/l_W = \beta^2 - \beta - 1 = 2\cos(\pi/7)$ a selfsimilar one-dimensional tiling is obtained with scaling factor β satisfying the equation $\beta^3 = 2\beta^2 + \beta - 1$.

The model presented in the following is an interpretation of words as planar patterns. The alphabet is $\Sigma = \{a_i, b_i, \bar{b}_i, c_i, d_i, \tilde{a}_i, \tilde{b}_i, \tilde{\bar{b}}_i, \tilde{c}_i, \tilde{d}_i, (,)\}$ with $i \in Z_{14}$. It contains two brackets and letters of type t_i and $\tilde{t}_i$ with $t = a, b, c, d$ which represent tiles with the same

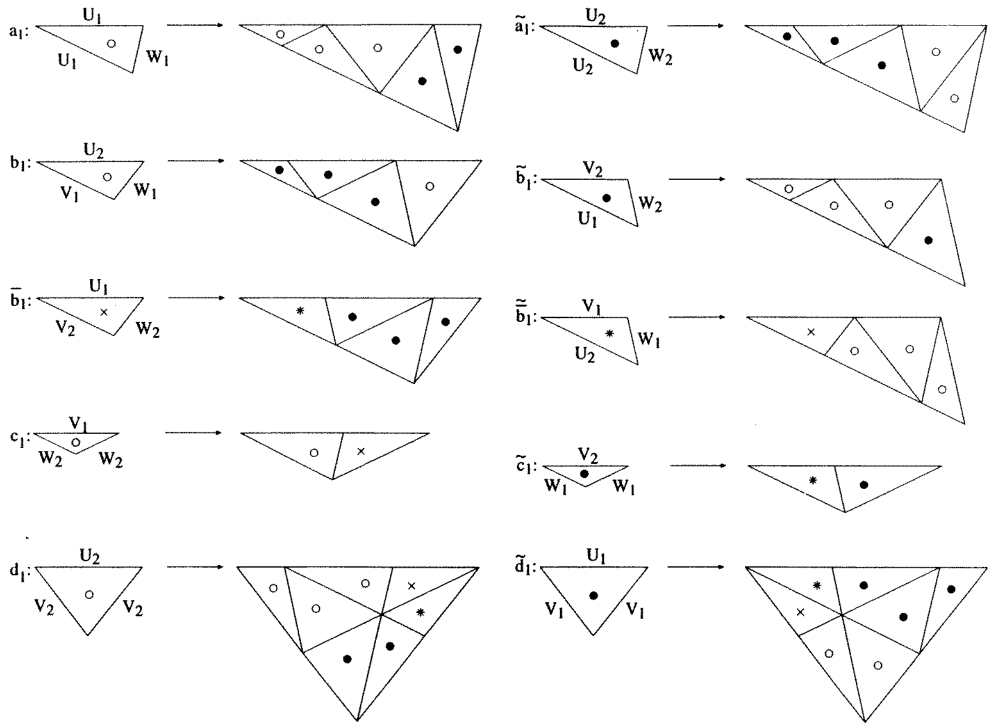

Figure 1. The first inflation step in the derivation of the patterns.

shape but they are distinguished by a colour. The tiles b and $\bar{b}$ have also the same shape but a different colouring. Therefore, we need two colours for t_i and $\tilde{t}_i$ and two additional colours for $\bar{b}_i$ and $\tilde{\bar{b}}_i$.

In figure 1 we can see the tiles represented by letters with subindex 1. The oriented tile with subindex i is obtained by rotation of $\pi(i-1)/7$ through the left-most vertex. Colours are represented by different markings on the tiles.

Every element belonging to Σ representing a tile can be used as an axiom. We take for instance $\omega = c_1$.The set of production rules is:

$$
\begin{aligned}
&a_i \longmapsto (\Phi[a_i]) = (\Phi[\tilde{b}_i]\tilde{a}_{i+4}) \qquad \tilde{a}_i \longmapsto (\Phi[\tilde{a}_i]) = (\Phi[b_i]a_{i-4}) \\
&b_i \longmapsto (\Phi[c_i]) = (\tilde{c}_{i+6}\tilde{b}_{i-6}\tilde{d}_{i+6}d_{i-5}) \qquad \tilde{b}_i \longmapsto (\Phi[\tilde{b}_i]) = (c_i b_{i+6} d_i \tilde{d}_{i-3}) \\
&\bar{b}_i \longmapsto (\Phi[\bar{b}_i]) = (\tilde{b}_{i+3}\tilde{d}_{i+1}\Phi[\tilde{c}_i]) \qquad \tilde{\bar{b}}_i \longmapsto (\Phi[\tilde{\bar{b}}_i]) = (b_{i-3}d_{i+5}\Phi[c_{i+6}]) \\
&c_i \longmapsto (\Phi[c_i]) = (a_i\bar{b}_{i-6}) \qquad \tilde{c}_i \longmapsto (\Phi[\tilde{c}_i]) = (\tilde{a}_{i-6}\tilde{\bar{b}}_i) \\
&d_i \longmapsto (\Phi[d_i]) = (\Phi[\tilde{\bar{b}}_{i-6}]\tilde{\bar{b}}_{i-5}\tilde{a}_{i+3}\tilde{d}_{i-4}) \qquad \tilde{d}_i \longmapsto (\Phi[\tilde{d}_i]) = (\Phi[\bar{b}_i]\bar{b}_{i-1}a_{i+5}d_{i+4}) \\
&) \longmapsto) \qquad (\longmapsto (.
\end{aligned}
\tag{2}
$$

The Coxeter group $H^{(7)} = \langle R_1, R_2/(R_1R_2)^7 = R_1^2 = R_2^2 = e\rangle$ contains two reflections R_1 and R_2 and a seven-fold rotation R_1R_2. The generators of $H^{(7)}$ acts on the symbols representing tiles as follows

$$
\begin{aligned}
R_1(x_i) &= \tilde{x}_{11-i} \qquad R_2(x_i) = \tilde{x}_{9-i} \\
R_1(y_i) &= \tilde{y}_{3-i} \qquad R_2(y_i) = \tilde{y}_{1-i}
\end{aligned}
\tag{3}
$$

where $x = a, b, \bar{b}$ and $y = c, d$. In this way the inflation represented by Φ satisfies $R_k \Phi^m = \Phi^m R_k$ for $k = 1, 2$ and $m \geqslant 1$.

Consider the following word sequence for a part of a pattern:

$$c_1, (a_1 \bar{b}_9), ((c_1 b_7 d_1 \tilde{d}_{12})(\tilde{b}_{12} \tilde{d}_{10} \tilde{a}_3 \tilde{\bar{b}}_9)).$$

In the second word of the sequence, if two letters follow one another, the corresponding oriented triangles are glued facet to facet in a unique way. This first inflation step in the derivation of the patterns can be seen in figure 1. The word $((c_1 b_7 d_1 \tilde{d}_{12})(\tilde{b}_{12} \tilde{d}_{10} \tilde{a}_3 \tilde{\bar{b}}_9))$ represents a part of the pattern where the scaled tile $(c_1 b_7 d_1 \tilde{d}_{12})$ is glued facet to facet with $(\tilde{b}_{12} \tilde{d}_{10} \tilde{a}_3 \tilde{\bar{b}}_9)$.

As one can see in successive patterns generations, the edges of the basic triangles do not inflate following the substitutional sequences obtained from (1). However, it is possible to give a colouring to the edges in order to get the correct inflations.

We label the edges as L_i where i stands for a colour and $L = U, V, W$. If we choose anticlockwise orientation in the patterns the following production rules are obtained:

$$\begin{array}{lll} U_1 \longmapsto \phi[U_1] = W_2 U_2 V_1 & V_1 \longmapsto \phi[V_1] = V_2 U_1 & W_1 \longmapsto \phi[W_1] = U_2 \\ U_2 \longmapsto \phi[U_2] = V_2 U_1 W_1 & V_2 \longmapsto \phi[V_2] = U_2 V_1 & W_2 \longmapsto \phi[W_2] = U_1. \end{array} \tag{4}$$

After projection of L_i into L, we get the correct words representing the inflated edges. Only L_1 and L_2 are common edges of two adjacent tiles. Therefore, the fact that the tilings are facet to facet is equivalent to the fact that $\phi^n(L_1)$ and $\phi^n(L_2)$ are mirror words after projection of L_j into L. The inflation rules for the edges can be useful in order to link these models with the non-commutative crystallography in dimension two ([7] and references therein), where the generators of free groups are associated with directed segments or paths.

The problem of how to know whether or not the diffraction pattern of a selfsimilar tiling consists of Bragg peaks has been solved by Bombery and Taylor in one dimension [8]. They found that the scaling factor should be a Pisot–Vijayaraghavan number. This is an algebraic integer greater than one with all its conjugates (the remaining roots of its minimal polynomial) strictly less than one in absolute value. Later these results have been generalized to several planar patterns [9]. Recently, necessary and sufficent conditions have been determined for a wave vector $\boldsymbol{q}$ to be in the Bragg spectrum. In [10] it is shown that the pattern has Bragg peaks only if the scaling factor of the pattern is a PV number.

It is possible to give another geometric description of the patterns presented in this work along the lines of [9, 4] that allows us to write recursion relations for the numerical computation of the diffraction spectra. Bragg peaks will be found due to the fact that β is a PV number.

References

[1] Senechal M 1995 *Quasicrystals and Geometry* (Cambridge: Cambridge University Press)
[2] García-Escudero J and Kramer P 1993 *J. Phys. A: Math. Gen.* **26** L1029
[3] García-Escudero J 1995 *J. Phys. A: Math. Gen.* **28** 5207
[4] García-Escudero J 1996 *J. Phys. Soc. Japan* **65**
[5] Nischke K P and Danzer L 1995 A construction of inflation rules based on n-fold symmetry *Disc. Comput. Geom.* to appear
[6] Rozenberg G and Salomaa A 1980 *The Mathematical Theory of L Systems* (New York: Academic)
[7] Kramer P 1995 *J. Phys. A: Math. Gen.* **28** 6405
[8] Bomberi E and Taylor J E 1986 *J. Physique Coll.* C **3** 19
[9] Godrèche C and Luck J M 1989 *J. Stat. Phys.* **55** 1
[10] Gähler F and Klitzing R 1995 The diffraction pattern of selfsimilar tilings *Proc. NATO ASI 'The Mathematical Foundations of Long Range: Aperiodic Order'* to appear